

Compressive Spectral Estimation with Single-Snapshot ESPRIT: Stability and Resolution

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Abstract

In this paper Estimation of Signal Parameters via Rotational Invariance Techniques (ESPRIT) is developed for spectral estimation with single-snapshot measurement. Stability and resolution analysis with performance guarantee for Single-Snapshot ESPRIT (SS-ESPRIT) is the main focus.

In the noise-free case *exact* reconstruction is guaranteed for any arbitrary set of frequencies as long as the number of measurement data is at least twice the number of distinct frequencies to be recovered. In the presence of noise and under the assumption that the true frequencies are separated by at least two times Rayleigh's Resolution Length, an explicit error bound for frequency reconstruction is given in terms of the dynamic range and the separation of the frequencies. The separation and sparsity constraint compares favorably with those of the leading approaches to compressed sensing in the continuum.

Keywords: ESPRIT, spectral estimation, stability, resolution, compressed sensing

1 Introduction

Suppose a signal $y(t)$ consists of linear combinations of s Fourier components from the set

$$\{e^{-2\pi i\omega_j t} : \omega_j \in \mathbb{R}, j = 1, \dots, s\}.$$

Suppose the external noise $\varepsilon(t)$ is present in the received signal

$$y^\varepsilon(t) = y(t) + \varepsilon(t), \quad y(t) = \sum_{j=1}^s x_j e^{-2\pi i\omega_j t}. \quad (1)$$

The problem of spectral estimation is to recover the frequency support set $\mathcal{S} = \{\omega_1, \dots, \omega_s\}$ and the corresponding amplitudes $x = [x_1, \dots, x_s]^T$ from a finite data sampled at, say, $t = 0, 1, 2, \dots, M \in \mathbb{N}$. Because of the nonlinear dependence of the signal $y(t)$ on frequency, the main difficulty of spectral estimation lies in identifying \mathcal{S} . The amplitudes x can be easily recovered by solving least squares once \mathcal{S} is known.

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Denote (with a slight abuse of notation) $y = [y_k]_{k=0}^M$, $\varepsilon = [\varepsilon_k]_{k=0}^M$ and $y^\varepsilon = y + \varepsilon \in \mathbb{C}^{M+1}$, with $y_k = y(k)$, $y_k^\varepsilon = y^\varepsilon(k)$ and $\varepsilon_k = \varepsilon(k)$. Let

$$\phi^M(\omega) = [1 \ e^{-2\pi i \omega} \ e^{-2\pi i 2\omega} \ \dots \ e^{-2\pi i M\omega}]^T \in \mathbb{C}^{M+1} \quad (2)$$

be the imaging vector of size $M + 1$ at the frequency ω and define

$$\Phi^M = [\phi^M(\omega_1) \ \phi^M(\omega_2) \ \dots \ \phi^M(\omega_s)] \in \mathbb{C}^{(M+1) \times s}.$$

The single-snapshot formulation of spectral estimation takes the form

$$y^\varepsilon = \Phi^M x + \varepsilon. \quad (3)$$

In addition to the nonlinear dependence of Φ^M on the unknown frequencies, with the sampling times $t = 0, 1, 2, \dots, M \in \mathbb{N}$, one can only hope to determine frequencies on the torus $\mathbb{T} = [0, 1)$ with the natural metric

$$d(\omega_j, \omega_l) = \min_{n \in \mathbb{Z}} |\omega_j + n - \omega_l|.$$

A key unit of frequency separation is Rayleigh's Resolution Length (RL), the distance between the center and the first zero of the sinc function $\sin(\pi\omega M)/(\pi\omega)$, namely, $1 \text{ RL} = 1/M$.

1.1 Single-Snapshot ESPRIT (SS-ESPRIT)

In this paper, to circumvent the gridding problem, we reformulate the spectral estimation problem (3) in the form of *multiple measurement vectors* suitable for the application of Estimation of Signal Parameters via Rotational Invariance Techniques (ESPRIT) [15, 17].

Most state-of-the-art spectral estimation methods ([16, 2, 20] and references therein) assume many snapshots of array measurement as well as statistical assumptions on measurement noise. Below we present a stability and resolution analysis for a *deterministic, single-snapshot* formulation of ESPRIT.

Fixing a positive integer $1 \leq L < M$, we form the Hankel matrix

$$H = \text{Hankel}(y) = \begin{bmatrix} y_0 & y_1 & \dots & y_{M-L} \\ y_1 & y_2 & \dots & y_{M-L+1} \\ \vdots & \vdots & \vdots & \vdots \\ y_L & y_{L+1} & \dots & y_M \end{bmatrix}. \quad (4)$$

It is straightforward to verify that $\text{Hankel}(y)$ with $y = \Phi^M x$ admits the Vandermonde decomposition

$$H = \Phi^L X (\Phi^{M-L})^T, \quad X = \text{diag}(x_1, \dots, x_s) \quad (5)$$

with the Vandermonde matrix

$$\Phi^L = \begin{bmatrix} 1 & 1 & \dots & 1 \\ e^{-2\pi i \omega_1} & e^{-2\pi i \omega_2} & \dots & e^{-2\pi i \omega_s} \\ (e^{-2\pi i \omega_1})^2 & (e^{-2\pi i \omega_2})^2 & \dots & (e^{-2\pi i \omega_s})^2 \\ \vdots & \vdots & \vdots & \vdots \\ (e^{-2\pi i \omega_1})^L & (e^{-2\pi i \omega_2})^L & \dots & (e^{-2\pi i \omega_s})^L \end{bmatrix}.$$

Let H_1 and H_2 be two sub-matrices of H consisting, respectively, of the first and last L rows of H . Clearly we have as before

$$H_1 = \Phi^{L-1} X (\Phi^{M-L})^T, \quad (6)$$

$$H_2 = \Phi^{L-1} \Lambda X (\Phi^{M-L})^T, \quad \Lambda = \text{diag}(e^{-2\pi i \omega_1}, \dots, e^{-2\pi i \omega_s}) \quad (7)$$

which can be rewritten as

$$H_1 = \Phi^{L-1} Y, \quad H_2 = \Phi^{L-1} \Lambda Y, \quad (8)$$

$$Y \equiv X (\Phi^{M-L})^T \in \mathbb{C}^{s \times (M-L+1)}. \quad (9)$$

Since Y has full (row) rank, $Y Y^\dagger = I$ where Y^\dagger denotes the pseudo-inverse of Y . Hence from (8) we have

$$H_2 = H_1 \Psi \quad (10)$$

with $\Psi = Y^\dagger \Lambda Y$ implying that $\{e^{-i2\pi\omega_1}, \dots, e^{-i2\pi\omega_s}\}$ is the set of nonzero eigenvalues of the unknown $(M-L+1) \times (M-L+1)$ matrix Ψ .

Theorem 1. *For the Hankel matrices H_1, H_2 given above,*

$$\Psi = H_1^\dagger H_2$$

is a rank- s solution to eq. (10).

Proof. Since $H_1 H_1^\dagger$ is the identity map on the range of H_1 , it suffices to prove $\text{Range}(H_1) = \text{Range}(H_2) = \text{Range}(\Phi^{L-1})$ which would follow from $\text{Rank}(\Phi^{M-L}) = s$.

On the other hand, we have $\text{Rank}(\Phi^{L-1}) = s$ if $L \geq s$ and $\omega_k \neq \omega_l, \forall k \neq l$. This is because $s \times s$ square submatrix Φ_s of Φ^L is a square Vandermonde matrix whose determinant is given by

$$\det(\Psi_s) = \prod_{1 \leq i < j \leq s} (e^{-i2\pi\omega_j} - e^{-i2\pi\omega_i}).$$

Clearly, Φ_s is invertible if and only if $\omega_i \neq \omega_j, i \neq j$. Hence $\text{Rank}(\Phi_s) = s$ which implies $\text{Rank}(\Phi^{L-1}) = s$. □

SS-ESPRIT is based on the following observation.

Theorem 2. *For the Hankel matrices H_1 and H_2 given above, let Ψ be any rank- s solution to $H_2 = H_1 \Psi$. Then $\{e^{-i2\pi\omega_1}, \dots, e^{-i2\pi\omega_s}\}$ is the set of nonzero eigenvalues of Ψ .*

Remark 1. *Theorem 2 implies that the number of measurement data $(M+1) \geq 2s$ suffices to guarantee exact reconstruction.*

Proof. From (8) we have

$$\Phi^{L-1} \Lambda Y = \Phi^{L-1} Y \Psi$$

and hence

$$\Lambda Y = Y \Psi \quad (11)$$

since Φ^{L-1} has full column rank. Using (9) and transposing (11) we obtain

$$\Phi^{M-L} X \Lambda = \Psi^T \Phi^{M-L} X$$

implying

$$\Phi^{M-L} \Lambda = \Psi^T \Phi^{M-L} \quad (12)$$

since X is diagonal, full rank and commutes with Λ . Eq. (12) means that the columns of Φ^{M-L} are the eigenvectors of the matrix Ψ^T with the diagonal entries of Λ as the corresponding eigenvalues. \square

Theorems 1 and 2 motivate the following reconstruction procedure in the case of noisy data.

Let $H^\varepsilon = \text{Hankel}(y^\varepsilon) = H + E$ where $E = \text{Hankel}(\varepsilon)$. Extracting H_1^ε and H_2^ε analogously from H^ε we have

$$H_1^\varepsilon = H_1 + E_1, \quad H_2^\varepsilon = H_2 + E_2$$

where E_1 and E_2 are two sub-matrices of E consisting, respectively, of the first and last L rows of E .

Let the SVD of H_1^ε be written as

$$H_1^\varepsilon = \left[\underbrace{U_1^\varepsilon}_{L \times \tau} \quad \underbrace{U_2^\varepsilon}_{L \times (L-\tau)} \right] \underbrace{\text{diag}(\sigma_1^\varepsilon, \sigma_2^\varepsilon, \dots, \sigma_s^\varepsilon, \sigma_{s+1}^\varepsilon, \dots)}_{L \times (M-L+1)} \left[\underbrace{V_1^\varepsilon}_{(M-L+1) \times \tau} \quad \underbrace{V_2^\varepsilon}_{(M-L+1) \times (M-L+1-\tau)} \right]^\star$$

with the singular values $\sigma_1^\varepsilon \geq \sigma_2^\varepsilon \geq \sigma_3^\varepsilon \geq \dots \geq \sigma_L^\varepsilon$. Let $\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \dots \sigma_s > 0$ be the nonzero singular values of H_1 . Without loss of generality, we assume $L \leq M - L + 1$ or equivalently $2L \leq M + 1$.

The number of frequencies s may be estimated when there is a significant spectral gap. For instance, according to [1], the spectral norm $\|E_1\|_2$ of a random Hankel matrix from a zero mean, independently and identically distributed (i.i.d.) sequence of a finite variance is on the order of $\sqrt{M \log M}$ for $M \gg 1$ while σ_s for well-separated frequencies is $\mathcal{O}(M)$ (see next section). Hence by Weyl's theorem [19]

$$|\sigma_j^\varepsilon - \sigma_j| \leq \|E_1\|_2, \quad j = 1, 2, \dots, L \quad (13)$$

the sparsity s can be easily estimated based on the singular value distribution of H^ε . Indeed, a spectral gap emerges because $\sigma_j^\varepsilon \leq \|E_1\|_2$, $\forall j \geq s + 1$ and $\sigma_s^\varepsilon \geq \sigma_s - \|E_1\|_2$.

Suppose the sparsity s is known and set $\tau = s$. Let $\mathcal{P}_s = U_1^\varepsilon (U_1^\varepsilon)^\star$ denotes the orthogonal projection onto the singular subspace of the s largest singular values. Consider the equation

$$\mathcal{P}_s H_2^\varepsilon = \mathcal{P}_s H_1^\varepsilon \Psi^\varepsilon, \quad (14)$$

equivalent to

$$U_1^\varepsilon \Sigma_s^\varepsilon (V_1^\varepsilon)^\star \Psi^\varepsilon = \mathcal{P}_s H_2^\varepsilon, \quad \Sigma_s^\varepsilon = \text{diag}(\sigma_1^\varepsilon, \sigma_2^\varepsilon, \dots, \sigma_s^\varepsilon) \quad (15)$$

Eq. (15) can then be solved as

$$\hat{\Psi} = \hat{H}_1^\dagger H_2^\varepsilon \quad (16)$$

$$\hat{H}_1 = U_1^\varepsilon \Sigma_s^\varepsilon (V_1^\varepsilon)^\star = \mathcal{P}_s H_1^\varepsilon \quad (17)$$

with rank- s $\hat{\Psi}$. Eq. (16)-(17) defines the main steps of Single-Snapshot ESPRIT (SS-ESPRIT). The rest is to find the nonzero eigenvalues of Ψ^ε and retrieve the frequencies from these eigenvalues.

2 Stability analysis

First we have

$$\begin{aligned}\|\hat{\Psi} - \Psi\|_2 &\leq \|(\hat{H}_1^\dagger - H_1^\dagger)H_2^\varepsilon\|_2 + \|H_1^\dagger(H_2^\varepsilon - H_2)\|_2 \\ &\leq \|\hat{H}_1^\dagger - H_1^\dagger\|_2\|H_2^\varepsilon\|_2 + \|H_1^\dagger\|_2\|E_2\|_2.\end{aligned}\tag{18}$$

Suppose at first $\|E_1\|_2 < \sigma_s$ (to be justified later) so that by Weyl's theorem $\sigma_s^\varepsilon > 0$ and $\text{Rank}(\hat{H}_1) = \text{Rank}(H_1)$. Wedin's inequality ([19], Theorem III.3.8) asserts that

$$\|\hat{H}_1^\dagger - H_1^\dagger\|_2 \leq \frac{1 + \sqrt{5}}{2} \|\hat{H}_1^\dagger\|_2 \|H_1^\dagger\|_2 \|\hat{H}_1 - H_1\|_2\tag{19}$$

First let us estimate $\|\hat{H}_1 - H_1\|_2$. We have

$$\begin{aligned}\|\hat{H}_1 - H_1\|_2 &= \|\mathcal{P}_s H_1^\varepsilon - H_1\|_2 \\ &\leq \|\mathcal{P}_s H_1^\varepsilon - H_1^\varepsilon\|_2 + \|H_1^\varepsilon - H_1\|_2 \\ &= \|(I - \mathcal{P}_s)H_1^\varepsilon\|_2 + \|E_1\|_2\end{aligned}$$

where $\mathcal{P}_s^\perp = I - \mathcal{P}_s$ is the projection onto the “noise” subspace of H_1^ε . Hence

$$\|\mathcal{P}_s^\perp H_1^\varepsilon\|_2 = \sigma_{s+1}^\varepsilon = \sigma_{s+1}^\varepsilon - \sigma_{s+1} \leq \|E_1\|_2$$

by Weyl's theorem (13). Therefore Wedin's bound (19) becomes

$$\|\hat{H}_1^\dagger - H_1^\dagger\|_2 \leq 2\|\hat{H}_1^\dagger\|_2 \|H_1^\dagger\|_2 \|E_1\|_2\tag{20}$$

and consequently the bound (18) becomes

$$\|\hat{\Psi} - \Psi\|_2 \leq \|H_1^\dagger\|_2 (2\|\hat{H}_1^\dagger\|_2 \|H_2^\varepsilon\|_2 \|E_1\|_2 + \|E_2\|_2) \equiv \eta.\tag{21}$$

Next we use the discrete Ingham inequalities to estimate

$$\|H_1^\dagger\|_2 = \sigma_s^{-1}, \quad \|\hat{H}_1^\dagger\|_2 = (\sigma_s^\varepsilon)^{-1}, \quad \|H_2^\varepsilon\|_2 = \sigma_1^\varepsilon.$$

The discrete Ingham inequalities are extension of the continuum version first proved in [12] (see also [22]).

Theorem 3. [14] *Let N be any integer. If \mathcal{S} satisfies the separation condition*

$$\delta = \min_{j \neq l} d(\omega_j, \omega_l) > \frac{1}{N} \left(1 - \frac{2\pi}{N}\right)^{-\frac{1}{2}}$$

then for any $z \in \mathbb{C}^s$

$$\frac{\|\Phi^N z\|_2^2}{\|z\|_2^2} \geq N \left(\frac{2}{\pi} - \frac{2}{\pi N^2 \delta^2} - \frac{4}{N} \right).\tag{22}$$

Moreover, when N is even

$$\frac{\|\Phi^N z\|_2^2}{\|z\|_2^2} \leq N \left(\frac{4\sqrt{2}}{\pi} + \frac{\sqrt{2}}{\pi N^2 \delta^2} + \frac{3\sqrt{2}}{N} \right)\tag{23}$$

and when N is odd

$$\frac{\|\Phi^N z\|_2^2}{\|z\|_2^2} \leq (N+1) \left(\frac{4\sqrt{2}}{\pi} + \frac{\sqrt{2}}{\pi(N+1)^2\delta^2} + \frac{3\sqrt{2}}{N+1} \right). \quad (24)$$

By the Vandermonde decomposition (6) for H_1 , Theorem 3 with $N = L-1, M-L$, immediately implies the following.

Corollary 1. *Under the separation condition*

$$\delta > \max \left\{ \frac{1}{L-1} \left(1 - \frac{2\pi}{L-1} \right)^{-\frac{1}{2}}, \frac{1}{M-L} \left(1 - \frac{2\pi}{M-L} \right)^{-\frac{1}{2}} \right\} \quad (25)$$

we have

$$\sigma_s \geq \frac{2x_{\min}}{\pi} \left(L-1 - \frac{1}{(L-1)\delta^2} - 2\pi \right)^{1/2} \left(M-L - \frac{1}{(M-L)\delta^2} - 2\pi \right)^{1/2} \quad (26)$$

$$\sigma_1 \leq \frac{4\sqrt{2}x_{\max}}{\pi} \left(L + \frac{1}{4L\delta^2} + \frac{3\pi}{4} \right)^{1/2} \left(M-L+1 + \frac{1}{4(M-L+1)\delta^2} + \frac{3\pi}{4} \right)^{1/2} \quad (27)$$

where

$$x_{\min} = \min_j \{|x_j|\}, \quad x_{\max} = \max_j \{|x_j|\}.$$

Remark 2. *For a fixed M , the right hand side of (26) can be maximized by $L = \lceil \frac{M+1}{2} \rceil$, the largest integer not greater than $\frac{M+1}{2}$, under which the separation condition (25) becomes*

$$\delta > \ell \equiv \frac{2}{M} \left(1 - \frac{4\pi}{M} \right)^{-\frac{1}{2}} \quad (28)$$

and hence the bounds (26)-(27) become

$$\sigma_1 \leq \frac{x_{\max}M2\sqrt{2}}{\pi} \left(1 + \frac{1}{M^2\delta^2} + \frac{2}{M} + \frac{3\pi}{2M} \right) \quad (29)$$

$$\sigma_s \geq \frac{x_{\min}M}{\pi} \left(1 - \frac{4}{M^2\delta^2} - \frac{4\pi}{M} \right). \quad (30)$$

Finally to justify the assumption $\sigma_s > \|E_1\|_2$ it suffices to have

$$\|E_1\|_2 < \frac{x_{\min}M}{\pi} \left(1 - \frac{4}{M^2\delta^2} - \frac{4\pi}{M} \right) \quad (31)$$

under (28) (which renders the right hand side positive).

By (29)-(30) and Weyl's theorem, we obtain

$$\|H_1^\dagger\|_2 \leq \frac{\pi}{x_{\min}M} \left(1 - \frac{4}{M^2\delta^2} - \frac{4\pi}{M} \right)^{-1} \quad (32)$$

$$\|\hat{H}_1^\dagger\|_2 \leq \left(\frac{x_{\min}M}{\pi} \left(1 - \frac{4}{M^2\delta^2} - \frac{4\pi}{M} \right) - \|E_1\|_2 \right)^{-1} \quad (33)$$

$$\|H_2^\varepsilon\|_2 \leq \frac{x_{\max}M2\sqrt{2}}{\pi} \left(1 + \frac{1}{M^2\delta^2} + \frac{2}{M} + \frac{3\pi}{2M} \right) \quad (34)$$

which lead to the corresponding bound on η via (21).

Summarizing the preceding analysis, we have the following theorem

Theorem 4. Let $\rho = \delta M$ be the minimum separation in the unit of RL. Under the separation condition (28), or equivalently

$$\rho > 2 \left(1 - \frac{4\pi}{M}\right)^{-\frac{1}{2}}, \quad (35)$$

and

$$\|E_1\|_2 < \frac{x_{\min} M}{\pi} \left(1 - \frac{4}{M^2 \delta^2} - \frac{4\pi}{M}\right)$$

we have

$$\|\hat{\Psi} - \Psi\|_2 \leq \|H_1^\dagger\|_2 (2\|\hat{H}_1^\dagger\|_2 \|H_2^\varepsilon\|_2 \|E_1\|_2 + \|E_2\|_2) \equiv \eta$$

with an upper bound given by (32)-(34). In particular, for $M \gg 1$, η has the asymptotic

$$\eta \approx \frac{\pi}{x_{\min}(1 - 4\rho^{-2})} \left[\frac{4\sqrt{2}(1 + \rho^{-2})x_{\max}}{(1 - 4\rho^{-2})x_{\min} - \pi\|E_1\|_2/M} \frac{\|E_1\|_2}{M} + \frac{\|E_2\|_2}{M} \right].$$

As noted before, the spectral norm of a random Hankel matrix from a zero mean, i.i.d. sequence of a finite variance is on the order of $\sqrt{M \log M}$ [1]. Therefore for i.i.d. noise the error bound in Theorem 4 tends to zero like $\sqrt{\log M/M}$ with a constant depending on the dynamic range x_{\max}/x_{\min} and the minimum separation $\rho > 2$ in the unit of RL.

Now we are ready to use Elsner's theorem ([19], Theorem IV.1.3) to conclude

$$\mu_H(\hat{\Psi}, \Psi) \leq \left(\|\hat{\Psi}\|_2 + \|\Psi\|_2\right)^{1 - \frac{1}{M-L+1}} \|\hat{\Psi} - \Psi\|_2^{\frac{1}{M-L+1}} \quad (36)$$

where

$$\mu_H(\hat{\Psi}, \Psi) = \max \left\{ \max_i \min_j |\hat{\lambda}_i - \lambda_j|, \max_j \min_i |\hat{\lambda}_i - \lambda_j| \right\}$$

is the Hausdorff Metric (HM) of the two sets of eigenvalues in question. Bound (36) can be made more concrete by using Theorem 4 and the fact $\|\Psi\|_2 = 1$:

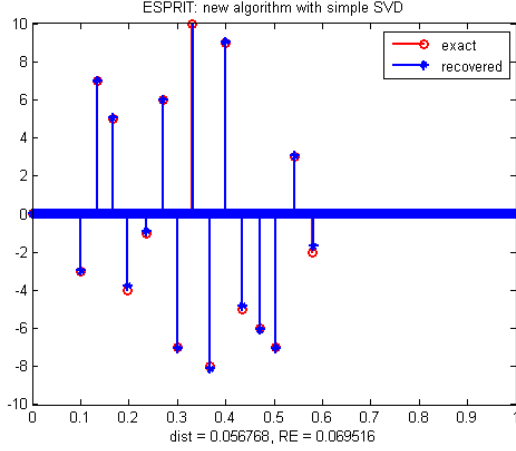
$$\mu_H(\hat{\Psi}, \Psi) \leq \left(2 + \eta\right)^{1 - \frac{1}{M-L+1}} \eta^{\frac{1}{M-L+1}} \quad (37)$$

3 Conclusion

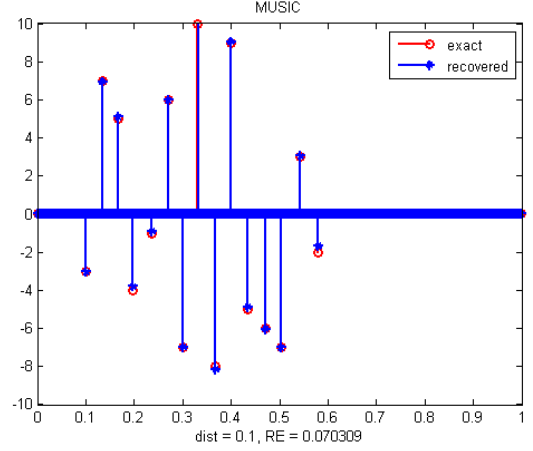
In conclusion, we have given performance guarantees for SS-ESPRIT. In particular, for noiseless measurement with $M+1 \geq 2s$, Theorems 1 and 2 guarantee exact recovery for any subset $\mathcal{S} \subset [0, 1]$ of s frequencies. For noisy measurement, Theorem 4 guarantees noise stability under the separation condition

$$\rho > 2 \left(1 - \frac{4\pi}{M}\right)^{-\frac{1}{2}}$$

in the unit of RL. This separation and sparsity constraint compares favorably with those of other approaches to compressed sensing in the continuum which are at least 3-4 RL [3, 4, 7, 8, 10, 21].



(a) ESPRIT: $\mu_H(\hat{\mathcal{S}}, \mathcal{S}) = 0.057\text{RL}$.



(b) MUSIC: $\mu_H(\hat{\mathcal{S}}, \mathcal{S}) = 0.1\text{RL}$.

Figure 1: Reconstruction of 15 real-valued frequencies separated by 3-4 RL with 10% NSR.

Numerical simulation demonstrates stability to a significant level of noise. The noise ε is additive i.i.d. complex Gaussian, i.e. $\varepsilon \sim N(0, \nu^2 I) + iN(0, \nu^2 I)$ of various strength in terms of the Noise-to-Signal Ratio (NSR)

$$\text{NSR} = \mathbb{E}\{\|\varepsilon\|_2\}/\|y\|_2 = \nu\sqrt{2(M+1)}/\|y\|_2$$

where $M = 100$. The error metric the Hausdorff metric $\mu_H(\mathcal{S}, \hat{\mathcal{S}})$ between the exact \mathcal{S} and recovered $\hat{\mathcal{S}}$ sets of frequencies. We use two reconstruction methods: SS-ESPRIT analyzed above and MUSIC studied in [14] (see also [5, 6, 9, 13]) both of which employ the Hankel matrix (4) and the Vandermonde decomposition (5).

Fig. 1 shows an instance of reconstruction of 15 frequencies that are randomly distributed, separated by 3-4 RL and have real-valued amplitudes of dynamical range $x_{\max}/x_{\min} = 10$, from $M = 100$ measured data of 10% NSR. Both ESPRIT and MUSIC perform well with comparable accuracy.

For Fig. 2, the frequency set \mathcal{S} consists of 20 randomly selected frequencies separated by 2 – 3 RL, with randomly phased amplitudes x of equal strength (i.e. the dynamic range $x_{\max}/x_{\min} = 1$). A reconstruction is *successful* if $\mu_H(\hat{\mathcal{S}}, \mathcal{S}) \leq 1\text{RL}$.

Fig.2(a) shows the success rate for 100 independent trials versus NSR. Clearly a “phase transition” occurs at the threshold $\text{NSR} \approx 37\%$ beyond which the success rate begins to drop precipitously. The threshold NSR depends on the frequency spacings, the numbers of data and frequencies as well as the dynamic range.

Fig.2(b) shows $\mu_H(\mathcal{S}, \hat{\mathcal{S}})$, averaged over 100 independent trials, versus NSR and exhibits the same phase transition where the rapid growth of μ_H is due to reconstruction failure. Notably the average μ_H below the threshold does not exceed 0.2RL, much better than the success criterion of 1RL.

Again the performances of ESPRIT and MUSIC are comparable in Fig. 2 with the main difference being the speed of computation: SS-ESPRIT is about ten times faster than MUSIC in our simulation.

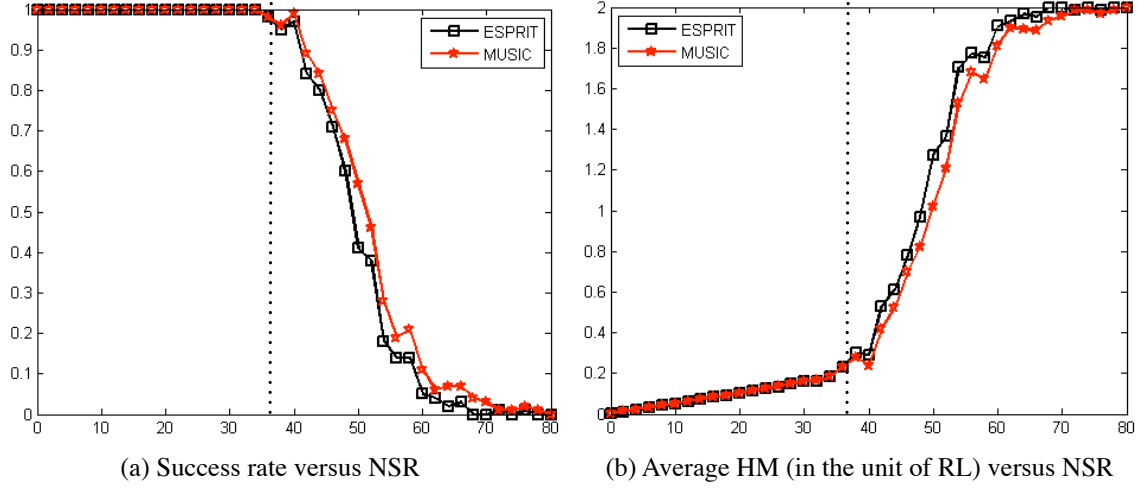


Figure 2: (a) Success rate and (b) average HM vs. NSR (in percentage) for separation of 2-3 RL

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